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Julia set of the function  $z \exp(z + \mu)$

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### Introduction

Let  $f_\mu$  be an entire transcendental function  $z \mapsto z \exp(z + \mu)$ , where  $\mu$  is a complex parameter. Put  $f_\mu^n = f_\mu \circ f_\mu^{n-1}$  for a positive integer  $n$ , where  $f_\mu^0$  means the identity mapping of the complex plane  $\mathbb{C}$ . The Julia set  $J_\mu$  of  $f_\mu$  is defined as the set of all points on  $\mathbb{C}$ , in any neighbourhood of every point of which the sequence  $\{f_\mu^n\}_{n=0}^\infty$  does not form a normal family.

Baker [1] proved the following theorem.

**Theorem** There exists a real value of the parameter  $\mu$  such that the Julia set  $J_\mu$  of  $f_\mu$  coincides with  $\mathbb{C}$ .

Jang [4] proved the following result by studying Baker's argument in detail : There are infinitely many positive real values of  $\mu$  with the property  $J_\mu = \mathbb{C}$ .

In this article, we study the distribution of values of  $\mu$  stated in the above result of Jang. Noting another result  $J_\mu \neq \mathbb{C}$  ( $-\infty < \mu < 2$ ) of Jang [4], we restrict the parameter  $\mu$  to the real value not less than 1.

§ 1 Values  $\mu_n$  and  $\mu^{(n)}$  of the parameter  $\mu$

Obviously the set of singular values of  $f : z \mapsto z \exp(z + \mu)$  consists of two values  $z = 0$  and  $z = f_\mu(-1)$ . The point  $z = 0$  is the only one finite transcendental singularity of the inverse function  $f_\mu^{-1}$  of  $f_\mu$  and this is fixed by  $f_\mu$ . The point  $z = f_\mu(-1)$  is the only one finite algebraic singularity of  $f_\mu^{-1}$ .

For a fixed value  $\mu$  of the parameter, we put

$$s_0(\mu) = -1 \quad \text{and} \quad s_n(\mu) = f_\mu(s_{n-1}(\mu)), \quad n \geq 1.$$

The sequence  $\{s_n(\mu)\}_{n=1}^\infty$  is the so-called orbit of the critical value  $z = f_\mu(-1)$  of  $f_\mu$  under the iteration of  $f_\mu$ . The behaviour of this orbit plays a very important role in the study of the bifurcation of Julia sets  $J_\mu$ . So, first we state some properties of  $s_n(\mu)$ .

Since the parameter  $\mu$  is real, every  $s_n(\mu)$  is negative and we have

$$(1) \quad s_n(\mu) = s_k(\mu) \exp \psi_{k, n-k}(\mu), \quad 0 \leq k \leq n-1,$$

where

$$(2) \quad \psi_{k, l}(\mu) = \sum_{j=k}^{k+l-1} (s_j(\mu) + \mu), \quad l \geq 1.$$

For an arbitrary real constant  $\alpha$ , we see

$$(3) \quad \lim_{\mu \rightarrow \infty} (s_1(\mu) + \alpha \mu) = -\infty.$$

As Jang [4] showed, (3) implies

$$(4) \quad \lim_{\mu \rightarrow \infty} s_n(\mu) = 0, \quad n \geq 2.$$

Evidently we see

$$(5) \quad \mu \leq -s_1(\mu) = \exp(-1 + \mu),$$

where the equality holds only for  $\mu = 1$ . In other words, the equation  $s_1(\mu) + \mu = 0$  in the unknown  $\mu$  has the only one root  $\mu_1 = 1$ . We see also that the equation  $s_1(\mu) + 1 = 0$  has the only one root  $\mu^{(1)} = 1$ . A simple calculation shows that  $s_2(\mu) + \mu = 0$  has the only one root  $\mu_2 = 1$  in the interval  $1 \leq \mu < \infty$  and that  $s_2(\mu) + \mu$  is positive for  $\mu > \mu_2$ . It is also easy to see that the equation  $\psi_{0,2}(\mu) = -1 + s_1(\mu) + 2\mu = 0$  has two roots  $\mu = 1$  and  $\mu = \mu^{(2)} (> 1)$  and  $\psi_{0,2}(\mu)$  is positive in the interval  $1 < \mu < \mu^{(2)}$  and is negative in the intervals  $0 < \mu < 1$  and  $\mu^{(2)} < \mu < \infty$ . Since we see

$$\psi_{0,2}(1 + \log 3) = -4 + 2(1 + \log 3) > 0,$$

the equation  $s_2(\mu) + 1 = -\exp \psi_{0,2}(\mu) + 1 = 0$  has the greatest root  $\mu^{(2)}$  greater than  $1 + \log 3$ .

For completeness of our discussion, we recall Jang's argument in [4] under a slight improvement. Since  $s_2(\mu^{(2)}) + 1 = 0$ , (5) implies

$$s_3(\mu^{(2)}) + \mu^{(2)} = s_1(\mu^{(2)}) + \mu^{(2)} < 0.$$

Hence (4) gives us the existence of the greatest root

$$\mu = \mu_3 (> \mu^{(2)}) \text{ of the equation } s_3(\mu) + \mu = 0.$$

Clearly  $s_3(\mu) + \mu$  is positive for  $\mu > \mu_3$ . Since  $s_3(\mu_3) = -\mu_3 < -\mu^{(2)} < -(1 + \log 3)$ , the equality (4) shows the existence of the greatest root  $\mu^{(3)} (> \mu_3)$  of

the equation  $s_3(\mu) + 1 = 0$ . Obviously  $s_3(\mu) + 1$  is positive for  $\mu > \mu^{(3)}$ .

We use  $\mu^{(3)}$  instead of  $\mu^{(2)}$  in the above observation and see the existence of the greatest root  $\mu_4 (> \mu^{(3)})$  of the equation  $s_4(\mu) + \mu = 0$  and the existence of the greatest root  $\mu^{(4)} (> \mu_4)$  of the equation  $s_4(\mu) + 1 = 0$ . It is easy to check that  $s_4(\mu) + \mu$  is positive for  $\mu > \mu_4$  and  $s_4(\mu) + 1$  is also positive for  $\mu > \mu^{(4)}$ .

Repeating the above procedure, we have a sequence of infinitely many values  $\mu_n$  and  $\mu^{(n)}$  of the parameter  $\mu$  such that

$$(6) \quad 1 = \mu_1 = \mu^{(1)} = \mu_2 < 1 + \log 3 < \mu^{(2)} < \mu_3 < \mu^{(3)} < \dots < \mu_n < \mu^{(n)} < \mu_{n+1} < \mu^{(n+1)} < \dots,$$

where

$$(7) \quad \begin{cases} s_n(\mu_n) + \mu_n = 0, & n \geq 1, \\ s_n(\mu) + \mu > 0 & \text{for } \mu > \mu_n, \quad n \geq 2 \end{cases}$$

and

$$(8) \quad \begin{cases} s_n(\mu^{(n)}) + 1 = 0, & n \geq 1, \\ s_n(\mu) + 1 > 0 & \text{for } \mu > \mu^{(n)}, \quad n \geq 2. \end{cases}$$

Remark Jang [4] states only that, for  $n \geq 3$ , the equation  $s_n(\mu) + \mu = 0$  has a root  $\mu_n (> \mu^{(n-1)})$  (not necessarily the greatest) and that the equation  $s_n(\mu) + 1 = 0$  has a root  $\mu^{(n)} (> \mu_n)$  (not necessarily the greatest).

## § 2 Distribution of the sequence $\{\mu_n\}_{n=1}^{\infty}$

First we prove the following proposition.

**Proposition 1** For values  $\mu^{(n)} (n \geq 2)$  of the parameter  $\mu$ , the  $n$  points  $s_k(\mu^{(n)})$ ,  $0 \leq k \leq n-1$ , are mutually distinct and are super-attractive  $n$ -th periodic points of  $f_{\mu^{(n)}}$ . Therefore, the Julia set of  $f_{\mu^{(n)}}$  does not coincide with  $\mathbb{C}$ .

**Proof** Suppose that there are integers  $k$  and  $\ell$  ( $0 \leq k < \ell \leq n-1$ ) with the property  $s_k(\mu^{(n)}) = s_\ell(\mu^{(n)})$ . Clearly  $s_k(\mu^{(n)}) = s_{k+q(\ell-k)}(\mu^{(n)})$  for any non-negative integer  $q$ . There is a positive integer  $p$  satisfying  $k + p(\ell - k) \leq n < k + (p+1)(\ell - k)$ . The sequence  $\{s_j(\mu^{(n)})\}_{j=k+p(\ell-k)}^{k+(p+1)(\ell-k)}$  containing  $s_m(\mu^{(n)})$  coincides with the sequence  $\{s_j(\mu^{(n)})\}_{j=k}^{\ell}$  and this shows the existence of such a  $j$  ( $k \leq j < \ell$ ) that  $s_j(\mu^{(n)}) = s_m(\mu^{(n)})$ . This contradicts (8). Thus  $n$  points  $s_k(\mu^{(n)})$  ( $0 \leq k \leq n-1$ ) are mutually distinct. Since  $f_{\mu^{(n)}}(-1) = 0$ , it is easy to see that these  $n$  points are super-attractive  $n$ -th periodic points of  $f_{\mu^{(n)}}$ .

On the value  $\mu_n (n \geq 3)$  of the parameter  $\mu$ , we can see that the point  $s_m(\mu_n)$  is a repulsive fixed point of  $f = f_{\mu_n}$ . To see this, we note (7) and (6) and have

$$f(s_m(\mu_n)) = f(-\mu_n) = -\mu_n$$

and

$$f'(s_m(\mu_n)) = f'(-\mu_n) = -\mu_n + 1 < -\log 3.$$

Thus  $s_n(\mu_n)$  is a repulsive fixed point of  $f$ . Hence, as Jang stated in [4], Baker's argument in [1], which was used to prove the theorem stated in the introduction of this article, leads us to the following result of Jang stated also in the introduction : The Julia set of  $f_{\mu_n}$  ( $n \geq 3$ ) coincides with  $\mathbb{C}$ . This is also proved in the following way. By Eremenko-Lyubich's theorem [2], the function  $f_{\mu_n}$  has no wandering domain and no Baker domain. Hence Sullivan's argument [5] implies  $J_{\mu_n} = \mathbb{C}$ .

Now we prove the following theorem.

Theorem 2  $\lim_{n \rightarrow \infty} \mu^{(n)} = \lim_{n \rightarrow \infty} \mu_n = \infty$ .

Proof By (6), it suffices to show  $\lim_{n \rightarrow \infty} \mu^{(n)} = \infty$ . Since the sequence  $\{\mu^{(n)}\}_{n=1}^{\infty}$  is increasing, we see the existence of  $\mu^{(\infty)} = \lim_{n \rightarrow \infty} \mu^{(n)} \leq \infty$ . Assume  $\mu^{(\infty)} < \infty$ . Clearly we have  $1 + \log 3 < \mu^{(\infty)}$  by (6) and  $-1 < s_n(\mu^{(\infty)}) < 0$  ( $n \geq 2$ ) by (8). Hence we have

$$\begin{aligned} s_{n+1}(\mu^{(\infty)}) / s_n(\mu^{(\infty)}) &= \exp(s_n(\mu^{(\infty)}) + \mu^{(\infty)}) \\ &> \exp(-1 + \mu^{(\infty)}) > 3 \end{aligned}$$

for every  $n (\geq 2)$ , which implies

$$-1 < s_{n+1}(\mu^{(\infty)}) < 3^{n-1} s_2(\mu^{(\infty)}).$$

The right hand side of this tends to  $-\infty$ , as  $n$  tends to infinity. This is a contradiction. Hence  $\mu^{(\infty)}$  must be infinity.

The above theorem can also be deduced from the following proposition.

Proposition 3  $\mu^{(n)} > 1 + \log(n+1)$  for  $n \geq 2$ .

Proof In the case  $n = 2$ , we have seen  $1 + \log 3 < \mu^{(2)}$  in (6). Hereafter, we consider the case  $n \geq 3$ .

Put  $y_1 = y_1(\mu) = -s_1(\mu)$ ,  $y_2 = y_2(\mu) = \psi_{0,n}(\mu) - s_1(\mu)$  and  $y_3 = y_3(\mu) = -(n-1) + n\mu$ . We see easily that the equation  $y_1 = y_3$  has two roots  $\mu = 1$  and  $\mu = \mu_*$  ( $> 1$ ) and that  $y_1 < y_3$  if and only if  $\mu$  is in the open interval  $1 < \mu < \mu_*$ .

In the case  $\mu_* \leq \mu^{(n-1)}$ , (6) implies  $\mu_* < \mu^{(n)}$ .

Consider the contrary case  $\mu^{(n-1)} < \mu_*$ . In this case, (6) and (8) give us  $s_k(\mu) + 1 > 0$  in  $\mu > \mu^{(n-1)}$  for  $2 \leq k \leq n-1$ . Hence we have

$$y_2 - y_3 = \sum_{j=0}^{n-1} (s_j(\mu) + \mu) - s_1(\mu) + (n-1) - n\mu > 0$$

for  $\mu > \mu^{(n-1)}$ . As was seen already, we have  $y_1 < y_3$  in the interval  $\mu^{(n-1)} < \mu < \mu_*$ . Hence we see  $y_1 < y_2$  in this interval. On the other hand, (3) and (4) imply

$$\lim_{\mu \rightarrow \infty} (y_2 - y_1) = \lim_{\mu \rightarrow \infty} \psi_{0,n}(\mu) = -\infty.$$

Since  $y_2(\mu_*) - y_1(\mu_*) = y_2(\mu_*) - y_3(\mu_*)$  is positive, the equation  $y_1 - y_2 = 0$  has a root greater than  $\mu_*$ .

As  $\mu^{(n)}$  is the greatest root of  $s_n(\mu) + 1 = 0$  and of

$$\psi_{0,n}(\mu) = y_2 - y_1 = 0, \text{ we see } \mu_* < \mu^{(n)}.$$

Thus we have always  $\mu_* < \mu^{(n)}$ . On the other hand, we have

$$\begin{aligned} y_1(1 + \log(n+1)) &= n+1 < 1 + n \log(n+1) \\ &= y_3(1 + \log(n+1)), \end{aligned}$$



which implies  $1 + \log (n + 1) < \mu_*$ . Therefore, we have

$$1 + \log (n + 1) < \mu^{(n)}$$

for  $n \geq 3$ . This is the required.

Remark By more careful observation, we can see

$$\mu^{(n)} > \begin{cases} 1 + \log (2n + 1) & n \geq 4, \\ 1 + \log (3n + 1), & n \geq 9, \\ 1 + \log (4n + 1), & n \geq 20 \end{cases}$$

and so on. The proofs of these may be omitted here.

We have also the following proposition.

Proposition 4  $\mu^{(3)} > 3$ .

Proof A direct calculation gives us

$$-74/10 < s_1(3) = -\exp 2 < -7.$$

Hence we see

$$s_2(3) = -\exp (5 + s_1(3)) > -\exp (-2) > -1/7$$

and

$$\begin{aligned} s_3(3) &= -\exp (8 + s_1(3) + s_2(3)) \\ &< -\exp (8 - 74/10 - 1/7) < -1. \end{aligned}$$

Since the value  $\mu^{(3)}$  is the greatest root of  $s_3(\mu) + 1 = 0$ , we have  $\mu^{(3)} > 3$  by (4).

Remark According to Sagawa,  $\mu^{(3)}$  lies between  $31/10$  and  $32/10$ .

### § 3 Repulsive periodic points of $f_\mu$ for some values of $\mu$

In the preceding section, we were concerned with the values  $\mu_n$  of the parameter  $\mu$ , each of which is the greatest root of the equation  $\psi_{n,1}(\mu) = s_n(\mu) + \mu = 0$ . In this section, we are concerned with the greatest root of the equation  $\psi_{n,k}(\mu) = 0$  for  $n \geq 3$  and  $k \geq 2$ . We see easily by (1) that, for this greatest root  $\mu$  of  $\psi_{n,k}(\mu) = 0$ ,  $s_{n+k}(\mu)$  is equal to  $s_n(\mu)$  so that  $s_n(\mu)$  is a periodic point of  $f_\mu$ .

Under the conditions  $n \geq 3$  and  $k \geq 2$ , we see  $\mu^{(n+k-2)} \geq \mu^{(3)}$  by (6). If  $\mu$  is not less than  $\mu^{(n+k-2)}$ , we see  $s_{n+k-2}(\mu) + 1 \geq 0$  and  $-1 < s_j(\mu) < 0$  for  $2 \leq j \leq n+k-3$ . Those are conclusions from (8). Hence we have

$$\begin{aligned} s_{n+k-3}(\mu) &= s_{n+k-2}(\mu) \exp(-s_{n+k-3}(\mu) - \mu) \\ &> s_{n+k-2}(\mu) \exp(1 - \mu) > -\exp(1 - \mu) \end{aligned}$$

for  $\mu \geq \mu^{(n+k-2)}$ . Similarly, for  $2 \leq j \leq n+k-4$ , we have

$$\begin{aligned} s_j(\mu) &> s_{j+1}(\mu) \exp(1 - \mu) \\ &> s_{n+k-3}(\mu) \exp((n+k-3-j)(1 - \mu)) \\ &> -\exp((n+k-2-j)(1 - \mu)) \end{aligned}$$

for  $\mu \geq \mu^{(n+k-2)}$ . Therefore, for  $2 \leq p \leq n+k-3$  and for  $\mu \geq \mu^{(n+k-2)}$ , we have

$$\begin{aligned} \sum_{j=p}^{n+k-3} s_j(\mu) &> -\sum_{j=p}^{n+k-3} \exp((n+k-2-j)(1 - \mu)) \\ &> -1/(\exp(\mu - 1) - 1). \end{aligned}$$

Proposition 4 and (6) imply

$$\sum_{j=p}^{n+k-3} s_j(\mu) > -1/((\exp 2) - 1) > -1/6$$

for  $2 \leq p \leq n+k-3$  and  $\mu \geq \mu^{(n+k-2)}$ . Hence we see

$$\begin{aligned} \psi_{0, n+k-2}(\mu^{(n+k-2)}) &= \sum_{j=0}^1 (s_j(\mu^{(n+k-2)}) + \mu^{(n+k-2)}) - (k-2)\mu^{(n+k-2)} \\ &= \sum_{j=2}^{n+k-3} s_j(\mu^{(n+k-2)}) + (n-2)\mu^{(n+k-2)} > 0. \end{aligned}$$

Here we recall  $\mu^{(n+k-2)}$  is a root of  $s_{n+k-2}(\mu) + 1 = 0$ , that is, a root of  $\psi_{0, n+k-2}(\mu) = 0$ . Hence the above inequality shows

$$(9) \quad \sum_{j=0}^1 s_j(\mu^{(n+k-2)}) + k\mu^{(n+k-2)} < 0.$$

Now we can prove the following proposition.

Proposition 5 For  $n \geq 3$  and  $k \geq 2$ , the equation  $\psi_{n,k}(\mu) = 0$  has the greatest root  $\mu = \mu_{n,k}$ , and  $\psi_{n,k}(\mu)$  is positive for  $\mu > \mu_{n,k}$ . In addition, the inequalities  $\mu^{(n+k-2)} < \mu_{n,k} < \mu^{(n+k-1)}$  hold.

Proof The inequality (9) shows

$$\begin{aligned} \psi_{n,k}(\mu^{(n+k-2)}) &= \sum_{j=n}^{n+k-1} (s_j(\mu^{(n+k-2)}) + \mu^{(n+k-2)}) \\ &< s_{n+k-2}(\mu^{(n+k-2)}) + s_{n+k-1}(\mu^{(n+k-2)}) + k\mu^{(n+k-2)} \\ &= s_0(\mu^{(n+k-2)}) + s_1(\mu^{(n+k-2)}) + k\mu^{(n+k-2)} < 0 \end{aligned}$$

by virtue of  $s_j(\mu) < 0$  and of  $s_{n+k-2}(\mu^{(n+k-2)}) = -1 = s_0(\mu^{(n+k-2)})$ . On the other hand, for  $\mu \geq \mu^{(n+k-1)}$ , we see

$$(10) \quad \psi_{n,k}(\mu) = \sum_{j=n}^{n+k-1} (s_j(\mu) + \mu) > -k + k\mu > 0$$

by (8) and (6). Hence there is the greatest root  $\mu_{n,k}$  of

the equation  $\psi_{n,k}(\mu) = 0$  such that  $\mu^{(n+k-2)} < \mu_{n,k} < \mu^{(n+k-1)}$ .  
Thus we have our proposition.

Using this proposition, we prove the following proposition.

**Proposition 6** For  $n \geq 3$  and  $k \geq 2$ , the points  $s_j(\mu_{n,k})$  ( $n \leq j \leq n+k-1$ ) are mutually distinct  $k$ -th periodic points of  $f_{\mu_{n,k}}$ .

**Proof** For simplicity, put  $\mu = \mu_{n,k}$  and  $f = f_\mu$ . As was stated at the beginning of this section,  $s_n(\mu)$  is equal to  $s_{n+k}(\mu)$ . So, it suffices to prove  $s_{n+j}(\mu) \neq s_{n+l}(\mu)$  for  $0 \leq j < l \leq k-1$ .

Assume  $s_{n+j}(\mu) = s_{n+l}(\mu)$  for  $0 \leq j < l \leq k-1$ . Then we see

$$s_{n+j}(\mu) = s_{n+l}(\mu) = f^{l-j}(s_{n+j}(\mu)) = s_{n+j}(\mu) \exp \psi_{n+j, l-j}(\mu),$$

which shows  $\psi_{n+j, l-j}(\mu) = 0$ . Proposition 5 shows that the greatest root of the equation  $\psi_{n+j, l-j}(\mu) = 0$  lies between  $\mu^{(n+l-2)}$  and  $\mu^{(n+l-1)}$ . So we have  $\mu < \mu^{(n+l-1)} \leq \mu^{(n+k-2)}$ . Since  $\mu = \mu_{n,k}$  is greater than  $\mu^{(n+k-2)}$  by Proposition 5, we have a contradiction. Therefore, we see  $s_{n+j}(\mu) \neq s_{n+l}(\mu)$  for  $0 \leq j < l \leq k-1$  and we have our proposition.

**Proposition 7** For  $n \geq 3$  and  $k \geq 2$ , the values  $\mu_{n,k}$  in Proposition 5 satisfy the following :

$$\mu^{(n+k-2)} < \mu_{3, n+k-3} < \mu_{4, n+k-4} < \dots < \mu_{n+k-2, 2} < \mu_{n+k-1} < \mu^{(n+k-1)}.$$

Proof First, as was stated in Proposition 5, we have

$$\psi_{n,k}(\mu_{n,k}) = \sum_{j=n}^{n+k-1} (s_j(\mu_{n,k}) + \mu_{n,k}) = 0.$$

Hence we see

$$\begin{aligned} \psi_{n+1,k-1}(\mu_{n,k}) &= \sum_{j=n+1}^{n+k-1} (s_j(\mu_{n,k}) + \mu_{n,k}) \\ &= \psi_{n,k}(\mu_{n,k}) - s_n(\mu_{n,k}) - \mu_{n,k} \\ &= -s_n(\mu_{n,k}) - \mu_{n,k}. \end{aligned}$$

By Proposition 5 and (6), we see  $\mu^{(n)} \leq \mu^{(n+k-2)} < \mu_{n,k}$ , which shows  $s_n(\mu_{n,k}) + 1 > 0$ . Hence (6) leads us to

$$\psi_{n+1,k-1}(\mu_{n,k}) = -s_n(\mu_{n,k}) - \mu_{n,k} < 1 - \mu_{n,k} < 0.$$

Therefore, we see by Proposition 5 that the greatest root

$\mu_{n+1,k-1}$  of the equation  $\psi_{n+1,k-1}(\mu) = 0$  is greater than  $\mu_{n,k}$ .

From this observation, we have

$$\mu^{(n+k-2)} < \mu_{3,n+k-3} < \mu_{4,n+k-4} < \dots < \mu_{n+k-2,2} < \mu^{(n+k-1)}.$$

Furthermore, since  $\mu_{n+k-1}$  is the greatest root of the

equation  $\psi_{n+k-1,1}(\mu) = s_{n+k-1}(\mu) + \mu = 0$ , we may put  $\mu_{n+k-1} = \mu_{n+k-1,1}$  in the notation used in Proposition 5. So, similarly to the above, we see easily  $\mu_{n+k-2,2} < \mu_{n+k-1} < \mu^{(n+k-1)}$ .

Thus we have our proposition.

Now we prove the following theorem.

**Theorem 8** Assume  $n \geq 3$  and  $k \geq 2$ . Then, for the values  $\mu_{n,k}$  of the parameter  $\mu$  obtained in Proposition 5, the Julia set of  $f_{\mu_{n,k}}$  coincides with  $\mathbb{C}$ .

Proof Proposition 6 shows that  $k$ -th periodic points  $s_j(\mu_{n,k})$  ( $n \leq j \leq n+k-1$ ) of  $f = f_{\mu_{n,k}}$  are mutually distinct. Suppose that there is a  $j$  ( $n \leq j \leq n+k-1$ ) with the property  $s_j(\mu_{n,k}) = -1$ . This means that the point  $-1$  is a  $k$ -th periodic point of  $f$  and we have  $s_k(\mu_{n,k}) = f^k(-1) = -1$ . This and (8) imply  $\mu_{n,k} \leq \mu^{(k)}$ . Proposition 5 leads us to a contradiction. Hence every point  $s_j(\mu_{n,k})$  ( $n \leq j \leq n+k-1$ ) is different from  $-1$ . The equation  $z \exp(z + \mu) = s_1(\mu) = -\exp(-1 + \mu)$  has the only one real root  $z = -1$  and hence the sequence  $\{s_j(\mu_{n,k})\}_{j=n}^{n+k-1}$  does not contain  $s_1(\mu_{n,k})$ . Therefore, the critical point  $s_1(\mu_{n,k})$  of  $f$  is a preperiodic point of  $f$ . In the same way as was stated after Proposition 1, Eremenko-Lyubich's theorem [2] and Sullivan's argument [4] give us the desired.

Remark In Fagella [3], we can find discussions about the same problem as ours.

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